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Necessary Conditions for Weighted Mean Convergence of Fourier Series in Orthogonal Polynomials*

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Necessary conditions are found for weighted mean convergence of Fourier series in orthogonal polynomials corresponding to measures $d\alpha$ with support $[-1, 1]$ for which $\alpha' > 0$ almost everywhere in $[-1, 1]$. Some additional properties of such orthogonal polynomials are also proved. © 1986 Academic Press, Inc.

Let $d\alpha$ be a finite positive Borel measure on the real line such that $\operatorname{supp}(d\alpha)$ is an infinite set and let $p_n(d\alpha)$ denote the corresponding orthonormal polynomials. For $f \in L^1_{d\alpha}$ let $S_n(d\alpha, f)$ denote the nth partial sum of the orthogonal Fourier expansion of f in $\{p_k(d\alpha)\}$, that is,

$$S_n(d\alpha, f) = \sum_{k=0}^{n} c_k p_k(d\alpha), \qquad c_k = \int_{-1}^{1} f p_k(d\alpha)\, d\alpha.$$

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It is well known [18] that $S_n(d\alpha, f) \to f$ in $L^2_{d\alpha}$ as $n \to \infty$ for every $f \in L^2_{d\alpha}$ if and only if the moment problem for $d\alpha$ possesses a unique solution, and the latter is certainly the case whenever $\operatorname{supp}(d\alpha)$ is bounded. The problem of weighted mean convergence of $S_n(d\alpha, f)$ to f in spaces different from $L^2_{d\alpha}$ has not yet been resolved with the exception of some specific orthogonal polynomial systems. For example, if $d\alpha$ and $d\beta$ are generalized Jacobi measures, then Badkov [4, 5] gave necessary and sufficient conditions for $L^p_{d\beta}$ convergence of $S_n(d\alpha, f)$ to f for every $f \in L^p_{d\beta}$. Badkov's results generalize earlier ones by Riesz [17], Pollard [14–16], Wing [20], Newman and Rudin [13], Muckenhoupt [9], Askey [1], and Badkov [3]. Orthogonal Hermite and Laguerre series were investigated in Askey and Wainger [2] and Muckenhoupt [10, 11]. In [12] one of us found necessary conditions for $L^p_{d\beta}$ convergence of $S_n(d\alpha, f)$ when $d\alpha$ belongs to the Szegö class [19], that is, when $\operatorname{supp}(d\alpha) = [-1, 1]$ and $\log \alpha'(\cos\theta) \in L^1[0, \pi]$. In the particular case when $d\alpha$ and $d\beta$ are generalized Jacobi measures, these conditions turn out to be sufficient as well [4]. In our recent papers [7, 8] we laid foundation to a theory of orhogonal polynomials that extends Szegö's theory when $\log \alpha'(\cos\theta) \in L^1[0, \pi]$ is replaced by the weaker condition that $\alpha' > 0$ a.e. in $[-1, 1]$. Our results enable us to prove the following generalization of Theorem 8.13 in [12, p. 154].

THEOREM 1. *Let α be such that* $\operatorname{supp}(d\alpha) = [-1, 1]$ *and $\alpha' > 0$ almost everywhere in $[-1, 1]$. Assume that p and q satisfy $0 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. Let u and w be Borel-measurable functions such that neither of them vanishes almost everywhere in $[-1, 1]$ and u is finite on a set with positive Lebesgue measure. Write $q' = q/(q-1)$ and*

$$v(x) = (\alpha'(x)\sqrt{1-x^2})^{1/2}.$$

Suppose that for every function $f \in L^1_{d\alpha}$ the inequality

$$\left(\int_{-1}^{1} |S_n(d\alpha, f)\, w|^p \, d\alpha\right)^{1/p} \leqslant C \left(\int_{-1}^{1} |fu|^q \, d\alpha\right)^{1/q} \tag{1}$$

holds for all integers $n \geqslant 0$ with a finite constant C independent of n and f (if $f(x) = 0$ and $u(x) = \infty$, then $f(x)\,u(x) = 0$ is to be taken in the integral on the right-hand side). Then $w \in L^p_{d\alpha}$, $u^{-1} \in L^{q'}_{d\alpha}$,

$$\left(\int_{-1}^{1} |w/v|^p \, \alpha'\right)^{1/p} < \infty, \tag{2}$$

and

$$\left(\int_{-1}^{1} |uv|^{-q'} \, \alpha'\right)^{1/q'} < \infty. \tag{3}$$

Here and in what follows, for $p=\infty$ the expression $(\int |g|^p\, d\alpha)^{1/p}$ means the $L^\infty_{d\alpha}$ norm of g. It may be worth pointing out that if $0<p<\infty$, $1<q<\infty$, and $p \leqslant q$ then in every known case (2) and (3) are also sufficient conditions for (1) to be satisfied (see, e.g., [4]).

Remark. We might as well allow that C in (1) depend on f, but then, by the Banach–Steinhaus theorem (cf., e.g., [6, Theorem 2.1.11 on p. 52]), it could be replaced by a constant independent of f.

Theorem 1 easily follows from Theorem 2 below, but first we have to prove a

LEMMA. *Let* $\operatorname{supp}(d\alpha)=[-1,1]$ *and* $\alpha'>0$ *a.e. in* $[-1,1]$. *For a given real c and a nonnegative integer n define the set* $B_{c,n}(d\alpha)$ *by*

$$B_{c,n}(d\alpha)=\{x: p_n^2(d\alpha,x)\,\alpha'(x)\sqrt{1-x^2}\geqslant c\}. \tag{4}$$

Then for every $c>2/\pi$

$$\lim_{n\to\infty} |B_{c,n}(d\alpha)|=0, \tag{5}$$

where $|E|$ *denotes the Lebesgue measure of the set E.*

Proof. Write

$$\Omega_n(x)=p_n^2(x)-2xp_n(x)\,p_{n-1}(x)+p_{n-1}^2(x).$$

Then

$$\Omega_n=(xp_n-p_{n-1})^2+(1-x^2)\,p_n^2,$$

so that

$$(1-x^2)\,p_n^2(x)\leqslant\Omega_n(x).$$

Therefore, if $D_{c,n}(d\alpha)$ is defined by

$$D_{c,n}(d\alpha)=\{x: \Omega_n(x)\,\alpha'(x)(1-x^2)^{-1/2}\geqslant c\}$$

then $B_{c,n}\subset D_{c,n}$. It was shown in [8, formula (10.3) after Theorem 10.1] that

$$\lim_{n\to\infty}\int_{-1}^{1}\left|\Omega_n(x)\,\alpha'(x)-\frac{2}{\pi}\sqrt{1-x^2}\right|dx=0.$$

Hence, for $c>2/\pi$

$$\lim_{n\to\infty}\int_{D_{c,n}}\left(\Omega_n(x)\,\alpha'(x)-\frac{2}{\pi}\sqrt{1-x^2}\right)dx=0$$

holds, so that

$$\lim_{n\to\infty}\left(c-\frac{2}{\pi}\right)\int_{D_{c,n}}\sqrt{1-x^2}\,dx=0,$$

from which

$$\lim_{n\to\infty}|D_{c,n}|=0 \qquad (c>2/\pi)$$

follows. Thus (5) must indeed hold.

THEOREM 2. *Let* $\operatorname{supp}(d\alpha)=[-1,1]$, $\alpha'>0$ *almost everywhere in* $[-1,1]$, *and suppose* $0<p\leqslant\infty$. *Put*

$$v(x)=(\alpha'(x)\sqrt{1-x^2})^{1/2}.$$

If g *is a Lebesgue-measurable function in* $[-1,1]$ *then*

$$\left(\int_{-1}^{1}|g/v|^p\right)^{1/p}\leqslant\sqrt{\pi}\,2^{\max\{1/p-1/2,0\}}\liminf_{n\to\infty}\left(\int_{-1}^{1}|gp_n(d\alpha)|^p\right)^{1/p}. \qquad (6)$$

In particular, if

$$\liminf_{n\to\infty}\left(\int_{-1}^{1}|gp_n(d\alpha)|^p\right)^{1/p}=0 \qquad (7)$$

then $g=0$ *a.e.*

Proof. First assume $0<p\leqslant 2$. Define r_n and h by

$$r_n=v^2p_n^2(d\alpha)$$

and

$$h=(|g|/v)^p,$$

respectively. Let

$$K=\liminf_{n\to\infty}\left(\int_{1}^{1}|gp_n(d\alpha)|^p\right)^{1/p}.$$

If $K=\infty$ then there is nothing to prove, so assume $K<\infty$. Then

$$\liminf_{n\to\infty}\int_{-1}^{1}hr_n^{p/2}=K^p$$

holds; therefore, if h_M is defined by

$$h_M(x) = \min\{h(x), M\}$$

for $M > 0$, then

$$\liminf_{n\to\infty} \int_{-1}^{1} h r_n^{p/2} \leqslant K^p \tag{8}$$

is satisfied as well. Fix $c > 2/\pi$. If $c > 2/\pi$. If $B_{c,n}$ is defined by (4) then (5) in the Lemma holds, and thus Theorem 13.2 of [8] implies

$$\lim_{n\to\infty} \int_{B_{c,n}} h_M r_n = 0.$$

Applying (11.4) in Theorem 11.1 of [8], we obtain

$$\lim_{n\to\infty} \int_{-1}^{1} h_M r_n = \frac{1}{\pi} \int_{-1}^{1} h_M. \tag{9}$$

Consequently,

$$\lim_{n\to\infty} \int_{[-1,1]\setminus B_{c,n}} h_M r_n = \frac{1}{\pi} \int_{-1}^{1} h_M \tag{10}$$

holds as well. On the other hand,

$$0 \leqslant r_n(x) < c$$

is satisfied for $x \in [-1, 1]\setminus B_{c,n}$, so that

$$0 \leqslant c^{p/2-1} r_n \leqslant r_n^{p/2} \qquad (x \in [-1, 1]\setminus B_{c,n})$$

holds. Thus by (8) we have

$$\liminf_{n\to\infty} \int_{[-1,1]\setminus B_{c,n}} h_M r_n \leqslant c^{1-p/2} K^p,$$

and combining this inequality with (10) we obtain

$$\int_{-1}^{1} h_M \leqslant \pi c^{1-p/2} K^p$$

for every $M > 0$ and $\varepsilon > 0$. Letting $M \to \infty$ here and applying Legesgue's Monotone Convergence Theorem, and then makin $c \to 2/\pi$, we can conclude that

$$\left(\int_{-1}^{1} h\right)^{1/p} \leqslant 2^{1/p-1/2} \sqrt{\pi}\, K,$$

and so the theorem follows for $0 < p \leqslant 2$. When $2 < p < \infty$ we can proceed as follows (the arguments below closely parallel those given in the proof of Theorem 7.32 in [12, pp. 138–139]). Keeping the previously established notation, from Hölder's inequality we obtain

$$\int_{-1}^{1} h_M r_n = \int_{-1}^{1} h_M^{(p-2)/p}(h_M^{2/p} r_n) \leqslant \left(\int_{-1}^{1} h_M\right)^{(p-2)/p} \left(\int_{-1}^{1} h_M r_n^{p/2}\right)^{2/p}.$$

Hence

$$\liminf_{n \to \infty} \int_{-1}^{1} h_M r_n \leqslant K^2 \left(\int_{-1}^{1} h_M\right)^{(p-2)/p},$$

which together with (9) implies

$$\left(\int_{-1}^{1} h_M\right)^{1/p} \leqslant \sqrt{\pi}\, K.$$

Letting $M \to \infty$, Lebesgue's Monotone Convergence Theorem entails

$$\left(\int_{-1}^{1} h\right)^{1/p} \leqslant \sqrt{\pi}\, K,$$

so that the theorem follows for $2 < p < \infty$ as well. Finally, assume $p = \infty$, and let $1 < q < \infty$. Clearly, we have

$$\left(\int_{-1}^{1} |f|^q\right)^{1/q} \leqslant 2^{1/q} \operatorname{ess.} \sup_{[-1,1]} |f| = 2^{1/q} \left(\int_{-1}^{1} |f|^p\right)^{1/p},$$

where the equation holds in view of the convention concerning the interpretation of the right-hand side for $p = \infty$. Therefore, inequality (6) with q replacing p implies

$$\left(\int_{-1}^{1} |g/v|^q\right)^{1/q} \leqslant \sqrt{\pi}\, 2^{1/q} \liminf_{n \to \infty} \left(\int_{-1}^{1} |g p_n(dx)|^p\right)^{1/p}$$
$$(p = \infty,\ 1 < q < \infty).$$

Making $q \to \infty$, inequality (6) follows for $p = \infty$ as well. Thus the proof of Theorem 2 is complete.

Proof of Theorem 1. For $n = 0$, inequality (1) implies

$$\left(\int_{-1}^{1} |w|^p\, dx\right)^{1/p} \left|\int_{-1}^{1} f\, dx\right| \leqslant C p_0^{\,2} \left(\int_{-1}^{1} |fu|^q\, dx\right)^{1/q} \tag{11}$$

for every $f \in L^1_{d\alpha}$. Since u is finite on a set of positive measure, we can find a Borel set E and a positive number N such that $d\alpha(E) > 0$ and $u(x) \leqslant N$ for $x \in E$. If f is the characteristic function of this set E then (11) shows that $w \in L^p_{d\alpha}$. If $1 < q \leqslant \infty$ then we can apply (11) with $f = (|u| + \varepsilon)^{-q'}$, where $\varepsilon > 0$ and $q' = q/(q-1)$; if we let $\varepsilon \to 0$, then $u^{-1} \in L^{q'}_{d\alpha}$ will follow by Fatou's lemma. If $q = 1$, then we apply (11) with $f = f_n$ being the characteristic function of the set where $|u^{-1}| > 1/n$; we obtain a contradiction unless $f_n = 0$ a.e. for large enough n; thus, we can conclude that $u^{-1} \in L^\infty_{d\alpha}$. Thus we have $u^{-1} \in L^{q'}_{d\alpha}$ for $1 \leqslant q \leqslant \infty$ (q is fixed), as claimed. Therefore $f = (fu)u^{-1} \in L^1_{d\alpha}$ also holds whenever $fu \in L^q_{d\alpha}$ $(1 \leqslant q \leqslant \infty)$.

Moreover, it follows from (1) that

$$\left(\int_{-1}^{1} |[S_n(f) - S_{n-1}(f)]\, w|^p\, d\alpha\right)^{1/p} \leqslant 2^{1+1/p} C \left(\int_{-1}^{1} |fu|^q\, d\alpha\right)^{1/q}$$

holds for $n \geqslant 1$ and $f \in L^1_{d\alpha}$. Hence we have

$$\left(\int_{-1}^{1} |p_n w|^p\, d\alpha\right)^{1/p} \left|\int_{-1}^{1} f p_n\, d\alpha\right| \leqslant 2^{1+1/p} C \left(\int_{-1}^{1} |fu|^q\, d\alpha\right)^{1/q} \tag{12}$$

for $n \geqslant 1$ and $f \in L^1_{d\alpha}$. Fix n and choose g such that

$$g p_n \geqslant 0 \qquad \text{and} \qquad |gu|^q = |p_n u^{-1}|^{q'}, \tag{13}$$

i.e.,

$$g = (|p_n|^{q'} u^{-q-q'})^{1/q} \qquad (g(x) = 0 \text{ if } u(x) = \infty).$$

Put $E = \{x \in [-1, 1] \colon g(x) \neq 0\}$. Let $E_k \subset E$ be a Borel set and h_k its characteristic function such that $h_k(x) \to 1$ as $k \to \infty$ for $x \in E$ and $gu \in L^q_{d\alpha}(E_k)$, i.e., $h_k gu \in L^q_{d\alpha}[-1, 1]$, for every k. Then $h_k g \in L^1_{d\alpha}[-1, 1]$ according to the last sentence of the preceding paragraph, i.e., (12) holds with $f = f_k = h_k g$. Noting that we have

$$f_k p_n = |f_k u|\, |p_n u^{-1}| = |f_k u|^q = |p_n u^{-1}|^{q'}$$

on E_k according to (13), the equality

$$\int_{E_k} f_k p_n\, d\alpha = \left(\int_{E_k} |f_k u|^q\, d\alpha\right)^{1/q} \left(\int_{E_k} |p_n u^{-1}|^{q'}\, d\alpha\right)^{1/q'}$$

holds. Thus (12) with $f = f_k$ implies

$$\left(\int_{-1}^{1} |p_n w|^p\, d\alpha\right)^{1/p} \left(\int_{E_k} |p_n u^{-1}|^{q'}\, d\alpha\right)^{1/q'} \leqslant 2^{1+1/p} C.$$

Making $k \to \infty$ and replacing E with $[-1, 1]$ in the second integral ($u^{-1} = 0$ outside E), we obtain

$$\left(\int_{-1}^{1} |p_n w|^p \, d\alpha\right)^{1/p} \left(\int_{-1}^{1} |p_n u^{-1}|^{q'} \, d\alpha\right)^{1/q'} \leqslant 2^{1+1/p} C$$

for all $n \geqslant 1$ ($q' = q/(q-1)$). By (7) in Theorem 2 this implies that

$$\sup_{n \geqslant 1} \left(\int_{-1}^{1} |p_n w|^p \, d\alpha\right)^{1/p} < \infty$$

and

$$\sup_{n \geqslant 1} \left(\int_{-1}^{1} |p_n u^{-1}|^{q'} \, d\alpha\right)^{1/q'} < \infty,$$

and now inequalities (2) and (3) follow from Theorem 2.

For orthogonal polynomials on the unit circle, the analogue of Theorem 2 can be derived without much difficulty from Theorem 2.1 of [8], and therefore one can easily formulate and prove a result similar to Theorem 1 for weighted mean boundedness of Fourier expansions in orthogonal polynomials on the unit circle. We lave leave the details to the reader.

We expect that Theorem 2 and the Lemma above will have further applications. In fact, we believe that these two statements will play a significant role in the extension of Szegö's theory we initiated in [7, 8]. An example is given by the following

THEOREM 3. *If* $\operatorname{supp}(d\alpha) = [-1, 1]$ *and* $\alpha' > 0$ *almost everywhere in* $[-1, 1]$ *then*

$$\sum_{k=0}^{\infty} |c_k p_k(d\alpha, x)| \tag{14}$$

either diverges almost everywhere or converges almost everywhere in $[-1, 1]$, *and in the latter case*

$$\sum_{k=0}^{\infty} |c_k| < \infty \tag{15}$$

holds as well.

Proof. By Theorem 2 with $p = 1$, we have

$$\liminf_{n \to \infty} \int_E |p_n(d\alpha)| \geqslant \frac{1}{\sqrt{2\pi}} \int_E v^{-1} > 0 \tag{16}$$

for every set E with positive Lebesgue measure. Now assuming that (14) converges on a set $E \subset [-1, 1]$, $|E| > 0$, one can apply (16) and the usual arguments used to prove the Denjoy–Lusin theorem on absolute convergence of trigonometric series [21, p. 232]. These give (15), from which the convergence of (13) almost everywhere in $[-1, 1]$ follows by Lebesgue's Monotone Convergence Theorem.

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